By successive application of isentropic flow relations along the stagnation point streamline in the region bounded by the stagnation point and the conical shock, the region bounded by the conical shock and the flare shock, and the region bounded by the flare shock and the point of interest, one can reduce Eq (2) to the following form:

$$G\left(\frac{s}{R_b}\right) = \left[\frac{(P_e/P_t)^{(5\gamma+19)/24\gamma_e}\,[1-(P_e/P_t)^{(\gamma-1)/\gamma}]^{0\,5}\,(r/R_b)^{1\,25}(s/R_b)}{\int_0^{s/R_b}(P/P_t)^{(5\gamma+19)/24\gamma_e}\,[1-(P/P_t)^{(\gamma-1)/\gamma}]^{0\,5}\,(r/R_b)^{1\,25}d(s/R_b)}\right]^{0\,2} \qquad (3)$$

A second heat-transfer calculation technique which is often used in the literature for data correlation is that suggested by Eckert [4] The corresponding Nusselt number correlation can be represented as follows:

$$Nu^* = 0\,0296\,(Pr^*)^{0\,333}\,(Re_s^*)^{0\,8} \qquad (4)$$

The two methods just outlined were utilized to obtain the correlation between predictions and data presented in Fig 3 Initially, Eqs (1) and (4) were used to predict Nusselt numbers for each of the instrumented sections Howevei, the results were nearly the same; thus an average value is presented Early time (high Reynolds number) data are subject to inaccuracies because of the small difference between the recovery and wall temperatures At a flight time of 100 sec when the local Reynolds number is approximately 10^6, boundary-layer transition appears to commence Therefore, subsequent data cannot be compared with the turbulent predictions

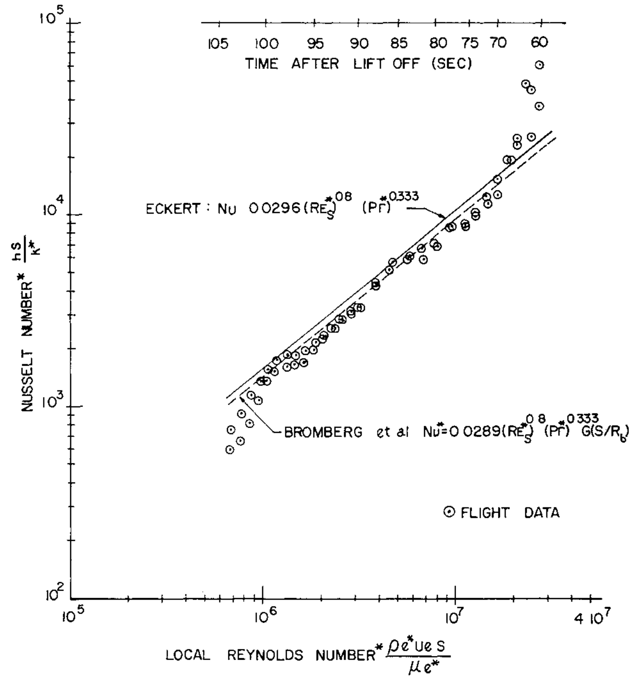

Fig 3 Comparison of observed and predicted Nusselt numbers

Conclusions

A correlation between two turbulent boundary-layer heat-transfer prediction techniques and recent flight data has been presented The Eckert correlation results in predicted Nusselt numbers which are approximately 10% greater than corresponding Bromberg values

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Transient Temperature Variation in a Thermally Orthotropic Cylindrical Shell

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An analysis is carried out to study the two-dimensional transient heat-transfer characteristics of a thin, anisotropic cylindrical shell An arbitrary heat input is assumed on the front face of the shell and the rest of the surfaces are assumed to be insulated

Giedt and Hornbaker[1] carried out an analysis to study the time-temperature history in a thermally anisotropic plate Such solutions have recently become necessary because of the increased use of such highly anisotropic materials as pyrolytic graphite for thermal protection

A similar solution applied to an infinite cylindrical shell (Fig 1) will be discussed here, subject to the same boundary conditions as those given by Giedt and Hornbaker The Fourier law of conduction for an anisotropic medium in cylindrical coordinates is given by

$$\frac{\partial v}{\partial t} = \kappa\left[\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r}\right] + \frac{\kappa_\theta}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} \qquad (1)$$

κ_r = thermal diffusivity in the r direction
κ_θ = thermal diffusivity in the θ direction
k_r = thermal conductivity in the r direction
v = temperature
t = time

Now let

$$r' = \frac{r}{(\kappa_r)^{1/2}} \qquad \theta' = \theta\left(\frac{\kappa_r}{\kappa_\theta}\right)^{1/2} \qquad (1a)$$

Substitution of these new variables in Eq (1) gives

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v_r}{\partial r'^2} + \frac{1}{r'}\frac{\partial v_r}{\partial r'} + \frac{1}{r'^2}\frac{\partial^2 v_\theta'}{\partial \theta^2} \qquad (2)$$

The appropriate initial and boundary conditions are

$$\frac{\partial v}{\partial r'} = 0 \text{ at } r' = \frac{a}{(\kappa)^{1/2}} = a' \qquad 0 < \theta < \theta_0 \qquad (2a)$$

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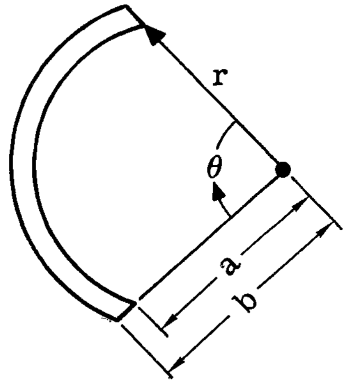

Fig 1 Coordinate system

$$\frac{\partial v}{\partial \theta'} = 0 \qquad \text{at } \theta' = 0,\ \theta' = \theta_0 \left(\frac{\kappa_r}{\kappa_\theta}\right)^{1/2} \qquad a < r < b \qquad (2b)$$

$$\frac{\partial v}{\partial r'} = g(\theta') \qquad \text{at } r' = \frac{b}{(\kappa\ )^{1/2}} = b' = \frac{(\kappa_r)^{1/2}}{k} f(\theta) \qquad (2c)$$

$$0 < \theta < \theta_0$$

$$v = 0 \qquad \text{at } t = 0 \qquad a < r < b,\ 0 < \theta < \theta_0 \qquad (2d)$$

Application of the Laplace transformation leads to

$$s\bar{v} = \frac{\partial^2 \bar{v}_r}{\partial r'^2} + \frac{1}{r'}\frac{\partial \bar{v}_r}{\partial r'} + \frac{1}{r'^2}\frac{\partial^2 \bar{v}_\theta}{\partial \theta'^2} \qquad (3)$$

then

$$g(\theta') = \sum_{n=0}^{\infty} B_n \cos(\beta\theta')$$

Henceforth, for clarity, we delete all prime notations that have to do with coordinate transformation:

$$\bar{v} = \sum_{n=0}^{\infty} \frac{B_n \cos\beta\theta \left\{K_\beta(s^{1/2}r) - \dfrac{K_\beta'(s^{1/2}a)}{I_\beta'(s^{1/2}a)} I_\beta(s^{1/2}r)\right\}}{s\left\{K_\beta'(s^{1/2}b) - \dfrac{K_\beta'(s^{1/2}a)}{I_\beta'(s^{1/2}a)} I_\beta'(s^{1/2}b)\right\}} \qquad (6)$$

where

$$I_\beta'(s^{1/2}r) = \frac{d}{dr}\{I_\beta(s^{1/2}r)\} = \frac{\beta}{r} I_\beta(s^{1/2}r) + s^{1/2} I_{\beta+1}(s^{1/2}r)$$

$$K_\beta'(s^{1/2}r) = \frac{d}{dr}\{K_\beta(s^{1/2}r) = \frac{\beta}{r} K_\beta(s^{1/2}r) - s^{1/2} K_{\beta+1}(s^{1/2}r)$$

also

$$I_\beta'(s^{1/2}a) \equiv \frac{d}{dr} I_\beta(s^{1/2}r)|_{=a} \qquad (6a)$$

$$I_\beta'(s^{1/2}b) \equiv \frac{d}{dr} I_\beta(s^{1/2}r)|_{=b} \qquad \text{etc} \qquad (6b)$$

On expanding Eq (6) by introducing identities (6a) and (6b), one gets

$$\bar{v} = \sum_{n=0}^{\infty} \left\{ B_n b \cos(\beta\theta) \frac{K_\beta(s^{1/2}r)[\beta I_\beta(s^{1/2}a) + s^{1/2}aI_{\beta+1}(s^{1/2}a)] - I_\beta(s^{1/2}r)[\beta K_\beta(s^{1/2}a) - s^{1/2}aK_{\beta+1}(s^{1/2}a)]}{s[K_1 I_1 - K_2 I_2]} \right\} \qquad (7)$$

The corresponding subsidiary conditions become

$$\frac{\partial \bar{v}}{\partial r'} = 0 \qquad \frac{\partial \bar{v}}{\partial \theta'} = 0 \qquad \frac{\partial \bar{v}}{\partial r'} = \frac{g(\theta')}{s} \qquad \text{etc}$$

Equation (3) is now a function of only two independent variables and its solution may be obtained by the method of separation of variables

Assuming that $\bar{v} = R(r')\theta(\theta')$, one has after substitution in Eq (3)

$$\theta''/\theta = -\beta^2$$

$$r'^2 \frac{d^2R}{dr'^2} + r'\frac{dR}{dr'} - (r'^2 s + \beta^2)R = 0 \qquad (4)$$

Then

$$\bar{v}(r',\theta') = [CI_\beta(s^{1/2}r') + DK_\beta(s^{1/2}r')] \times [A \sin\beta\theta' + B\cos\beta\theta'] \qquad (5)$$

From boundary condition (2b) $A \equiv 0$, and

$$\beta = \pi n/\theta_0' = \pi n/\theta_0(\kappa_\theta/\kappa_r)^{1/2} \qquad n = 0, 1, 2$$

From boundary condition (2a) one has

$$\frac{C}{D} = -\left.\frac{(d/dr')K_\beta(s^{1/2}r')}{(d/dr')I_\beta(s^{1/2}r')}\right|_{r=a}$$

where

$$K_1 = [\beta K_\beta(s^{1/2}b) - s^{1/2}bK_{\beta+1}(s^{1/2}b)]$$

$$I_1 = [\beta I_\beta(s^{1/2}a) + s^{1/2}aI_{\beta+1}(s^{1/2}a)]$$

$$I_2 = [\beta I_\beta(s^{1/2}b) + s^{1/2}I_{\beta+1}(s^{1/2}b)]$$

$$K_2 = [\beta K_\beta(s^{1/2}a) - s^{1/2}aK_{\beta+1}(s^{1/2}a)]$$

The procedure described for inverting Eq (7) is given in the Appendix The resulting equation is, in the original coordinate system,

$$v = \frac{(\kappa_r)^{1/2}}{k_r} A_0 \frac{2b^2 t}{b^2 - a^2} + \sum_{n=1}^{\infty}\left\{\pi b^2 \left[\frac{\frac{1}{2}(r/a)^2 + \ln(r/a) - \frac{1}{2}}{(b^2 - a^2)}\right] \frac{(\kappa_r)^{1/2}}{k} \times A_n \cos\left[\beta\theta\left(\frac{\kappa_r}{\kappa_\theta}\right)^{1/2}\right] \sum_{m=1}^{\infty} e^{-\kappa\alpha^2_m \beta t} F_\beta(a,b,\alpha_{\beta m})\right\} \qquad (8)$$

which may be shown formally to satisfy the differential Eq (1)

In the foregoing

$$\beta = \frac{\pi n}{\theta_0}\left(\frac{\kappa_\theta}{\kappa}\right)^{1/2} \qquad f(\theta) = \sum_{n=0}^{\infty} A_n \cos\left[\frac{n\pi}{\theta_0}\left(\frac{\kappa_\theta}{\kappa_r}\right)^{1/2}\right]$$

$$F_\beta(a,b,\alpha) = \frac{\{Y_\beta(\alpha_{m,\beta}r)[\beta J_\beta(\alpha_{m,\beta}b) - \alpha_{m,\beta}bJ_{\beta+1}(\alpha_{m,\beta}b)] - J_\beta(\alpha_{m,\beta}r)[\beta Y_\beta(\alpha_{m,\beta}b) - \alpha_{m,\beta}bY_{\beta+1}(\alpha_{m,\beta}b)]\}}{M(\alpha_{m\,\beta})} [\beta J_\beta(\alpha_{\beta\,m}a) - \alpha_{\beta\,m}aJ_{\beta+1}(\alpha_{\beta\,m}a)]^2$$

and from (2c)

$$\sum_{n=0}^{\infty} B_n \cos\beta\theta' \left\{ C\frac{d}{dr} I_\beta(s^{1/2}r')|_{r=b} + D\frac{d}{dr}K_\beta(s^{1/2}r')|_{r=b}\right\} = \frac{g(\theta')}{s}$$

$$M(\alpha_{m\,\beta}) = (\beta^2 - \alpha^2 b^2)[\beta J_\beta(\alpha_{\beta\,m}b) - bJ_{\beta+1}(\alpha_{\beta\,m}b)]^2 - (\beta^2 - \alpha^2 a^2)[\beta J_\beta(\alpha_{\beta\,m}a) - aJ_{\beta+1}(\alpha_{m\,\beta}a)]^2$$

Here $\alpha_{m\,\beta}$, $m = 1, 2,$ are positive real roots[3] of

$$[\beta Y_\beta(b\alpha) - \alpha bY_{\beta+1}(\alpha b)][\beta J_\beta(\alpha a) - \alpha aJ_{\beta+1}(\alpha a)] - [\beta Y_\beta(a\alpha) - \alpha aY_{\beta+1}(a\alpha)][\beta J_\beta(\alpha b) - \alpha bJ_{\beta+1}(\alpha b)] = 0 \qquad (9)$$

Through the use of recurrence formulas one can further simplify Eq (9) to yield

$$J_\beta'(x)Y_\beta'(kx) - J_\beta'(kx)Y_\beta'(x) = 0 \tag{10}$$

where $k = b/a$ and $x = a\alpha$ Some of the roots of this equation are given by Truell in Ref 4

Appendix

In Eq (7) let the denominator be called $s\Delta s$, the numerator N, and $B_n \cos\beta\theta \equiv A_n$, for short From the inversion theorem

$$v(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\,t}\bar{v}(s)ds \tag{A1}$$

Let $e^{\,t}\bar{v}(s) = E$ For $\beta \neq 0$, E has a simple pole at $s = 0$ and simple poles at $s = -\kappa\,\alpha^2{}_{m\,\beta}$ where $\alpha_{m\,\beta}$ are the roots (all real and simple) of

$$\begin{aligned}\Delta s = {} & [\beta K_\beta(s^{1/2}b) - s^{1/2}bK_{\beta+1}(s^{1/2}b)] \times \\ & [\beta I_\beta(s^{1/2}a) + s^{1/2}aI_{\beta+1}(s^{1/2}a)] - \\ & [\beta I_\beta(s^{1/2}b) + s^{1/2}bI_{\beta+1}(s^{1/2}b)] \times \\ & [\beta K_\beta(s^{1/2}a) - s^{1/2}aK_{\beta+1}(s^{1/2}a)] = 0\end{aligned} \tag{A2}$$

For $\beta \neq 0$ the residue at the pole $s = 0$ is

$$\lim_{s\to 0}\left\{s\frac{N}{s\Delta s}\right\} = \frac{(a/r)^\beta + (r/a)^\beta}{\beta[(b/a)^\beta - (a/b)^\beta]} \tag{A3}$$

where, for $x \to 0$, we have used the relations

$$I_\beta(x) \to (1/2^\beta\beta!)x^\beta$$

$$K_\beta(x) \to 2^{\beta-1}(\beta - 1)!x^{-\beta}\beta \neq 0$$

$$K_0(x) \to -\ln x$$

For $\beta = 0$ the residue at the pole $s = 0$ is

$$\lim_{s\to 0}\frac{d}{ds}\left\{e^{\,t}\frac{s^{1/2}aK_0(s^{1/2}r)I_1(s^{1/2}a) - s^{1/2}aI_0(s^{1/2}r)K_1(s^{1/2}a)}{abI_1(s^{1/2}b)K_1(s^{1/2}a) - abI_1(s^{1/2}a)K_1(s^{1/2}b)}\right\} \tag{A4}$$

Let $\mu = s^{1/2}$; then $d/ds = (1/2\mu)(d/d\mu)$ By differentiating Eq (A4) and taking the limit $s \to 0$, one has for the residue of $s = 0$

$$\frac{2bt}{b^2 - a^2} + b\,\frac{(r/a)^2/2 + \ln(r/a) - 1/2}{b^2 - a^2} \tag{A5}$$

To find residue at the pole $s = -\kappa_r\alpha^2{}_{m\,\beta}$, we need

$$s\left.\frac{d(\Delta s)}{ds}\right|_{s = -\kappa\,\alpha^2{}_{m\,\beta}} = \frac{1}{2}\,\mu\left.\frac{d\Delta}{d\mu}\right|_{\mu = i\alpha_{m\,\beta}\kappa_r{}^{1/2}} =$$

$$\frac{1}{2}\,i\alpha_{m\,\beta}\kappa_r{}^{1/2}\left[\begin{aligned} & 2\beta/\mu\{[\beta I_\beta(\mu a) + \mu a I_{\beta+1}(\mu a)] \times \\ & [\beta K_\beta(\mu b) - \mu b K_{\beta+1}(\mu b)] - \\ & [\beta I_\beta(\mu b) + \mu b I_{\beta+1}(\mu b)] \times \\ & [\beta K_\beta(\mu a) - \mu a K_{\beta+1}(\mu a)]\} + \\ & b[\beta I_\beta(\mu a) + \mu a I_{\beta+1}(\mu a)] \times \\ & [\mu b K_\beta(\mu b) + \beta K_{\beta+1}(\mu b)] - \\ & b[b\mu I_\beta(\mu b) - \beta I_{\beta+1}(\mu b)] \times \\ & [\beta K_\beta(\mu a) - \mu a K_{\beta+1}(\mu a)] + \\ & a[\mu a I_\beta(\mu a) - \beta I_{\beta+1}(\mu a)] \times \\ & [\beta K_\beta(\mu b) - \mu b K_{\beta+1}(\mu b)] - \\ & a[\mu a K_\beta(\mu a) + \beta K_{\beta+1}(\mu a)] \times \\ & [\beta I_\beta(\mu b) + \mu b I_{\beta+1}(\mu b)]\end{aligned}\right] \tag{A6}$$

From Eq (A2) one has

$$\frac{\beta K_\beta(\mu b) - \mu b K_{\beta+1}(\mu b)}{\beta K_\beta(\mu a) - \mu a K_{\beta+1}(\mu a)} = \frac{\beta I_\beta(\mu b) + \mu b I_{\beta+1}(\mu b)}{\beta I_\beta(\mu a) + \mu a I_{\beta+1}(\mu a)} = \rho$$

If the foregoing is abbreviated into $m/n = o/l$, respectively, Eq (A6) may be written

$$\begin{aligned}s\,[d(\Delta s)/ds] = {} & \tfrac{1}{2}i\alpha_{m\,\beta}\kappa^{1/2}\{bl[\mu b K_\beta(\mu b) + \beta K_{\beta+1}(\mu b)] - \\ & bn[b\mu I_\beta(\mu b) - \beta I_{\beta+1}(\mu b)] + am[\mu a I_\beta(\mu a) - \beta I_{\beta+1}(\mu a)] - \\ & ao[\mu a K_\beta(\mu a) + \beta K_{\beta+1}(\mu a)]\}\end{aligned} \tag{A7}$$

In addition, one has the identities

$$\begin{aligned} & I_\beta(x) = i^{-\beta}J_\beta(ix) && K_\beta(x) = (\pi/2)i^{\beta+1}H_\beta{}^{(1)}(ix) \\ & J_\beta(-y) = (-1)^\beta J_\beta(y) && H_\beta{}^{(1)}(-y) = (-1)^\beta H_\beta{}^{(1)}(y)\end{aligned} \tag{A8}$$

Then

$$\rho = \frac{\beta J_\beta(\alpha_{m,\beta}b) - bJ_{\beta+1}(\alpha_{m,\beta}b)}{\beta J_\beta(\alpha_{m\,\beta}a) - aJ_{\beta+1}(\alpha_{m\,\beta}a)}$$

or

$$\left[s\frac{d\Delta s}{ds}\right]_{s = -\alpha^2{}_{m\,\beta}} = \frac{1}{2}\rho\,(\beta^2 - \alpha^2{}_{m\,\beta}b^2) - \frac{1}{2\rho}(\beta^2 - \alpha^2{}_{m\,\beta}a^2) =$$

$$\frac{M(\alpha_{m,\beta})}{2[\beta J_\beta(\alpha_{m\,\beta}a) - aJ_{\beta+1}(\alpha_{m\,\beta}a)][\beta J_\beta(\alpha_{m\,\beta}b) - bJ_{\beta+1}(\alpha_{m\,\beta}b)]}$$

where

$$\begin{aligned}M(\alpha_{m\,\beta}) = {} & (\beta^2 - \alpha^2{}_{m\,\beta}b^2)[\beta J_\beta(\alpha_{m\,\beta}b) - bJ_{\beta+1}(\alpha_{m\,\beta}b)]^2 - \\ & (\beta^2 - \alpha^2{}_{m\,\beta}a^2)\,[\beta J_\beta(\alpha_{m\,\beta}a) - aJ_{\beta+1}(\alpha_{m\,\beta}a)]^2\end{aligned}$$

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Effect of Thermal Accommodation on Cylinder and Sphere Drag in Free Molecule Flow

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Nomenclature

C_D	=	drag coefficient
F,G	=	functions defined by Eqs (7) and (8)
I_0, I_1	=	modified Bessel functions of first kind, zero, and first order
M	=	Mach number
Q	=	heat transfer rate
Q^*	=	dimensionless heat transfer rate defined by Eq (6)
S	=	molecular speed ratio
T	=	freestream static temperature
T_R	=	ratio of surface to freestream static temperature
T_w	=	surface temperature
α	=	thermal accommodation coefficient
σ, σ'	=	coefficients of tangential and normal momentum transfer
γ	=	specific heat ratio
ρ	=	density of freestream

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